Math 275D Lecture 20 Notes

Daniel Raban

November 18, 2019

1 When is the Itô Integral Zero?

1.1 The Itô integral with respect to a stopping time

Let ω_0 be such that $B(\omega_0, t) = 0$ for all t. Then do we get $(\int_0^t f\,dB_s)(\omega)$ for all t? The issue is that $\int_0^T f\,dB_s$ is only L^2-unique.

When discussing the Itô integral, we constructed it as a limit of $I_T(f_n)$, where $f_n \to f$ are simple functions:

$$f_n = \sum a_k^{(n)}(\omega)\mathbb{1}_{[t_k^{(n)}, t_{k+1}^{(n)}]}(t), \qquad \int_0^t f_n\,dB_s = \sum a_k^{(n)} B(\Delta_k^{(n)}).$$

Then if $(\int_0^t f_n\,dB_s)(\omega_0) = 0$ for all t, then $(\int_0^t f\,dB_s)(\omega_0) = 0$.

Theorem 1.1. *Let $f(\omega, t)$ be $\Omega \times [0, T]$ measurable and adapted to $\mathcal{F}_t$. Let*

$$X(\omega, t) = \int_0^t f(\omega, s)\,dB_s,$$

and let ν be a stopping time such that $f(\omega, t) = 0$ if $t < \nu(\omega)$. Then $X(\omega, t) = 0$ if $t < \nu(\omega)$.

Example 1.1. Let

$$f(\omega, t) = \begin{cases} 1 & |B(\omega, t)| \geq 2 \\ 0 & \text{otherwise}, \end{cases} \qquad \nu(\omega) = \inf\{t : |B(\omega, t)| \geq 1\}$$

Then $X(\omega, t) = 0$ if $t < \nu(\omega)$.

We want to say something like

$$\int_0^t f\,dB_s = \int_0^{t \wedge \nu} f\,dB_s = \int_0^{t \wedge \nu} 0\,dB_s = 0,$$

But we do not have any definition for integrating with bounds determined by a random variable.

Lemma 1.1. *Let $f, g \in L^2$ be adapted, and let ν be a stopping time. If $f = g$ for $t \le \nu$, then $X(t) = Y(t)$ for $t \le \nu$. Here, $X(t) = \int_0^t f\,dB_s$, $Y(t) = \int_0^t g\,dB_s$.*

Proof. We know $f_n \to f$, where $f_n = \sum a_k^{(n)}(\omega)\mathbb{1}_{[t_k^{(n)}, t_{k+1}^{(n)}]}(t)$ are simple functions. We can choose these to be $t_k^{(n)} = k/2^n$. Define

$$\widetilde{f}_n = \sum a_k^{(n)}(\omega)\mathbb{1}_{[t_k^{(n)}, t_{k+1}^{(n)}]}(t)\mathbb{1}_{\{\nu \le t_k^{(n)}\}}.$$

Then $\widetilde{f}_n(\omega, t) = 0$ if $t \le \nu$. These are still simple functions, so we can compute the integrals of them; we have

$$\left(\int_0^t \widetilde{f}_n\,dB_s\right)(\omega, t) = 0 \qquad \text{if } t < \nu.$$

We need to show $\widetilde{f}_n \to f$ in $L^2(\Omega \times [0, T])$. Check this.

If we have $(\int \widetilde{f}_n\,dB_s)(\omega, t) = 0$ if $t < \nu$, then for fixed t, $(\int_0^t f\,dB_s)(\omega, t) = 0$ in the event $t < \nu$ (with probability 1). But this is not enough to say that $\int_0^t f\,dB_s = 0$ for all t a.s. But it holds at all rational t with probability 1. Since $I_t(f)$ is continuous in t, we get that this is 0 for all t with probability 1. □

1.2 Extending the integral to L^2_{loc}

Recall $\mathcal{H}^2 = \{f \in L^2(\Omega \times [0, T]) : f \text{ is adapted}\}$. Even some basic functions do not make it into this collection: $e^{B(s)^4} \notin L^2$. We must find some way to extend the integral to a more general class of functions. Define $L^2_{\text{loc}} = \{f : \mathbb{P}(\int_0^T |f(\omega, t)|^2\,dt < \infty) = 1, f \text{ is adapted}\}$. This class contains $\mathcal{H}^2$: if $\mathbb{E}[X] < \infty$, then $X < \infty$ a.s. This class even encompasses functions such as $e^{e^{e^{B(s)}}}$.

Why stop at 2? We can define L^p_{loc} similarly; then $L^p_{\text{loc}} \subseteq L^q_{\text{loc}}$ if $p > q$.

Example 1.2. Let $f(s) = (1 - s)^{-1/2}$; this is not random. Then what is $\int_0^1 f\,dB_s$? Define

$$Z_n = \int_{1-2^{-n}}^{1-2^{-n-1}} f\,dB.$$

Then $\mathbb{E}[Z_n] = Z_n$, and $\text{Var}(Z_n) = \|f\|_{L^2(1-2^{-n}, 1-2^{n-1})} = \log(2)$. The Z_n are independent, but they all have the same variance. So if we break the integral down into more and more pieces, there is no way we can make sense of this.